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Converse Ackermann Property and Minimal Negation*

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RESUMEN

Una lógica S tiene la Conversa de la Propiedad Ackermann (CAP) si no es posible derivar en S proposiciones necesitivas a partir de proposiciones no necesitivas. Mostramos en este artículo cómo introducir la negación mínima en las lógicas positivas con la CAP. Definimos semánticas relacionales ternarias para todas las lógicas consideradas.

ABSTRACT

A logic S has the Converse Ackermann Property (CAP) if non-necessitive propositions are not derivable in S from necessitive ones. We show in this paper how to introduce minimal negation in positive logics with the CAP. Relational ternary semantics are provided for all the logics considered in this paper.

I. INTRODUCTION

A positive logic with a truth constant t and a falsity constant F has the Converse Ackermann Property (CAP) if all the formulas of the form $(A\to B)\to C$ are unprovable whenever C does not contain $\to$ or t or F. The CAP can intuitively be interpreted as the non-derivability of necessitive propositions from non-necessitive ones (a formula A is *necessitive* if A is equivalent to one of the form $\Box B$ and $\Box B$, in its turn, is defined as $(B\to B)\to B$ in logics in which $\Box$ is not primitive (see Anderson and Belnap [1975 §4.3]).

The question about which systems do posses the CAP is first posed in Anderson & Belnap [1975, §8.12.], and in Méndez [1987] it is answered for implicative and for positive logics. Syntactically speaking, the solution roughly consists in restricting *Contraction*

$$(A\to(A\to B))\to(A\to B)$$

and *Assertion*

$$A\to((A\to B)\to B)$$

to the case in which B is an implicative formula (A is *implicative* iff A is of the form $B\to C$). Thus, logics with the CAP are contractionless logics. Actually, they are the natural bridge between strict contractionless logics and logics with contraction.

On the other hand, and turning to minimal negation, it is known that this form of introducing negation arose in the context of intuitionistic logic (see Johansson [1936], Kolmogorov [1967]). The idea was to add a propositional falsity constant F to what we will now name the positive fragment of intuitionistic logic. Next, since no specific axiom on F is added, the definition

$$\neg A =_{df} A\to F$$

is introduced. Then, it is clear that the positive logic itself provides the set of negation theorems. The resulting logic, minimal intuitionistic logic, can consequently be viewed as a definitional extension of positive intuitionistic logic.

But, what about negation in logics with the CAP? That is, which kind(s) of negation(s) is (are) compatible with the CAP? We briefly note the results we are aware of. In Méndez [1988] a sort of semiclassical negation, in Kamide [2002] a so-called "strong negation" is added to the positive logics of Méndez [1987]. Now, the aim of this paper is to define minimal negation (in the sense discussed above) within these systems. That is to say, we shall definitionally extend the positive logics of Méndez [1987] with F and study the resulting minimal negation in each one of them.

The structure of the paper is as follows. In §II, III, the positive logics of Méndez [1987] are summarily recalled. In §IV, F is added and the resulting logics are syntactically studied. In §V, we present the semantics, and in §VI, we prove the completeness theorem. In §VII, the truth constant t is introduced and semantic consistency and completeness are proved for the new logics. In §VIII, the relationship between t and F is studied. Finally, in §IX, a set of matrices is presented. This set grounds some claims made throughout the paper.

II. Positive Logics with the CAP

The sentential language has the binary connectives $\to$, $\wedge$, $\vee$ as primitive. The biconditional ($\leftrightarrow$) is introduced by definition in the customary way. The logics we are here concerned with are defined from the following set of axiom schemes and rules of inference:

A1. $A\to A$
A2. $(B\to C)\to((A\to B)\to(A\to C))$
A3. $(A\to B)\to((B\to C)\to(A\to C))$
A4. $(A\to(A\to(B\to C)))\to(A\to(B\to C))$

A5. $A\to((A\to(B\to C))\to(B\to C))$
A6. $A\to(A\to A)$
A7. $A\to (B\to A)$
A8. $(A\wedge B) \to A$ $(A\wedge B)\to B$
A9. $((A\to B)\wedge(A\to C))\to(A\to(B\wedge C))$
A10. $A\to(A\vee B)$ $B\to(A\vee B)$
A11. $((A\to C)\wedge(B \to C))\to((A\vee B)\to C)$
A12. $(A\wedge(B\vee C))\to((A\wedge B)\vee(A\wedge C))$

Rules: *Adjunction* (Adj.) (if ðA and ðB, then ð$A\wedge B$), *Modus Ponens* (MP) (if ð$A\to B$ and ðA, then ðB], *CAP Assertion* (CAP as.) (if ðA, then ð$(A\to (B\to C))\to(B\to C)$), and the *rule K* (K) (if ðA, then ð$B\to A$).

The logics are defined as follows. The logic T^{o}_{+} (positive Ticket entailment — cfr. Anderson & Belnap [1975] — with the CAP) is formulated with A1–A4, A8–A12, Adj. and MP. Other logics are defined as follows:

E^{o}_{+}: T^{o}_{+} plus CAP. As.
R^{o}_{+}: T^{o}_{+} plus A5.
RMO^{o}_{+} R^{o}_{+} plus A6.
$S4^{o}_{+}$: E^{o}_{+} plus K
I^{o}_{+}: T^{o}_{+} plus A7

If in all foregoing formulations we change (whenever present) A4, CAP as. and A5 for, respectively, *Contraction*

A4'. $(A\to(A\to B))\to(A\to B)$

the *Assertion rule* (as.r)

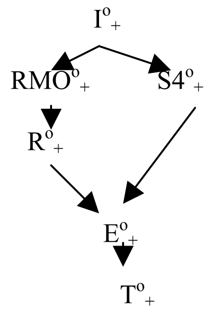
as.r: if ðA, then ð$(A\to B)\to B$

and *Assertion*

A5'. $A\to((A\to B)\to B)$

we get the formulations of the following positive logics, respectively: Ticket entailment, T_{+}, Entailment, E_{+}, Relevance logic, R_{+}, Relevance logic plus the mingle axiom, RMO_{+}, modal logic S4, $S4_{+}$, and intuitionistic logic, I_{+}. So, T^{o}_{+}, E^{o}_{+}, R^{o}_{+}, RMO^{o}_{+}, $S4^{o}_{+}$ and I^{o}_{+} are the restrictions with the CAP of the previously mentioned logics (see Anderson & Belnap [1975], Méndez [1987]).

The deductive relations these logics maintain with each other (which are exactly those maintained by their unrestricted counterparts) are summarized in the following diagram where the arrow stands for set inclusion:



III. SEMANTICS FOR POSITIVE LOGICS

Given a triple ⟨*K*, *O, R*⟩ where *K* is a non-empty set, $O \subseteq K$ and *R* a ternary relation on *K*, let us define the binary relation ≤, the quaternary relation R^2 and the five element relation R^3 by: for every *a ,b ,c , d*, *e* ∈ *K*:

d1. $a \leq b =_{\mathrm{df}} (\exists x \in O)\ Rxab$
d2. $R^2abcd =_{\mathrm{df}} (\exists x \in K)$ (*Rabx* and *Rxcd*)
d3. $R^3abcde =_{\mathrm{df}} (\exists x \exists y \in K)$ (*Rabx* and *Rxcy* and *Ryde*)

A T^o_+ model is a quadruple ⟨*K*, *O, R,* ö⟩ where *K* is a non-empty set, $O \subseteq K$ and *R* is a ternary relation on *K* satisfying the following conditions for every *a ,b ,c , d* ∈ *K*

P1. $a \leq a$
P2. ($a \leq b$ and *Rbcd*) ⇒ *Racd*
P3. $R^2abcd \Rightarrow (\exists x \in K)$ (*Rbcx* and *Raxd*)
P4. $R^2abcd \Rightarrow (\exists x \in K)$ (*Racx* and *Rbxd*)
P5. $R^2abcd \Rightarrow R^3abbcd$

Finally, ö is a evaluation relation from *K* to the sentences of the positive language satisfying the following conditions for each propositional variable *p*, any wff *A, B* and points *a*, *b* in *K*:

(i) (*a*ö *p* and $a \leq b$) ⇒ *b*ö*p*
(ii) *a*ö$A \wedge B$ iff *a*ö*A* and *a*ö*B*
(iii) *a*ö$A \vee B$ iff *a*ö*A* or *a*ö*B*
(iv) *a*ö $A \rightarrow B$ iff for all $b, c \in K$, (*Rabc* and *b*ö*A*) ⇒ *c*ö*B*

A is valid ($ö_{T^o_+}A$) iff *a*ö*A* for all $a \in O$ in all models.

Semantic for the remaining logics are defined from the following set of postulates:

P6. $Rabc \Rightarrow (\exists x \in O)\ R^2axbc$
P7. $R^2abcd \Rightarrow R^2bacd$
P8. $Rabc \Rightarrow a \leq c$ or $b \leq c$
P9. $(\exists x \in O)\ x \leq a$
P10. $Rabc \Rightarrow a \leq c$

In particular, we have (in correspondence to the axiomatic systems in §II): E^o_+ models, R^o_+ models, RMO^o_+ models, $S4^o_+$ models and I^o_+ models are the same as T^o_+ models but with the addition of the postulates P6, P7, P8, P9 and P10, respectively. Validity is similarly defined as in T^o_+.

These logics (and the accompanying semantics) are those defined in Méndez [1987] but only with this difference: we momentarily dispense with the truth constant *t* of Méndez [1987] (which is introduced in §VII below). It is then easy to prove, along the the lines of Méndez [1987], that *A* is valid iff *A* is a theorem for each one of these logics.

IV. The logics $T^o_{+,F}$, $E^o_{+.F}$, $R^o_{+,F}$, $RMO^o_{+,F}$, $S4^o_{+,F}$, $I^o_{+,F}$

The logics $T^o_{+,F}$, $E^o_{+,.F}$, $R^o_{+,F}$, $RMO^o_{+,F}$, $S4^o_{+,F}$, and $I^o_{+,F}$ are expansions with the propositional constant *F* of T^o_+, E^o_+, $R^o_{+,}$ RMO^o_+, $S4^o_+$, and I^o_+, respectively. We add to the sentential language of §II the propositional falsity constant *F* together with the

Definition : $\neg A =_{df} A \rightarrow F$

We begin by noting some characteristic theorems of the unrestricted positive logics (see, e.g., Anderson and Belnap [1975]):

(i) $(A\rightarrow(A\rightarrow C))\rightarrow(A\rightarrow C)$
(ii) $(A\rightarrow(B\rightarrow C))\rightarrow((A\rightarrow B)\rightarrow(A\rightarrow C))$
(iii) $(A\rightarrow B)\rightarrow((A \rightarrow(B\rightarrow C))\rightarrow(A\rightarrow C))$
(iv) $(A\rightarrow(B\rightarrow C))\rightarrow((A\wedge B)\rightarrow C)$
(v) $(A\rightarrow B)\rightarrow((A\rightarrow B)\rightarrow C)\rightarrow C)$
(vi) $(A\rightarrow((B\rightarrow D)\rightarrow C))\rightarrow((B\rightarrow D)\rightarrow(A\rightarrow C))$
(vii) $(B\rightarrow D)\rightarrow((A\rightarrow((B\rightarrow D)\rightarrow C))\rightarrow(A\rightarrow C)))$
(viii) $A\rightarrow((A\rightarrow C)\rightarrow C)$
(ix) $(A\rightarrow(B\rightarrow C))\rightarrow(B\rightarrow (A\rightarrow C))$
(x) $B\rightarrow((A\rightarrow(B\rightarrow C))\rightarrow(A\rightarrow C))$

Now, (i)-(iv) are theorems of T_+, (i)-(vii) of E_+, and (i)-(x), of R_+. It is easy to show that (i)-(x) are theorems of the respective logics with the CAP when C is restricted to an implicative formula (see Robles and Méndez [2002]). Therefore, they are also theorems of the corresponding definitionally extended logics when C is a negative formula (that is, an implicative formula of the form $A\to F$). Therefore, we have:

T1. $(A\to(A\to\neg B))\to(A\to\neg B)$
T2. $(A\to(B\to\neg C))\to((A\to B)\to(A\to\neg C))$
T3. $(A\to B)\to((A\to(B\to\neg C))\to(A\to\neg C))$
T4. $(A\to(B\to\neg C))\to((A\wedge B)\to\neg C)$
T5. $(A\to B)\to(((A\to B)\to\neg C)\to\neg C)$
T6. $(A\to((B\to C)\to\neg D))\to((B\to C)\to(A\to\neg D))$
T7. $(B\to C)\to((A\to((B\to C)\to\neg D))\to(A\to\neg D))$
T8. $A\to((A\to\neg C)\to\neg C)$
T9. $(A\to(B\to\neg C))\to(B\to(A\to\neg C))$
T10. $B\to((A\to(B\to\neg C))\to(A\to\neg C))$

Where T1-T4 are theorems of $T^{o}_{+,F}$, T1-T7, of $E^{o}_{+,F}$ and T1-T10, of $R^{o}_{+,F}$. On the other hand, we shall employ the following additional theorems:

(xi) $(A\to B)\to(A\to(A\to B))$
(xii) $(A\to B)\to(B\to(A\to B))$
(xiii) $B\to(A\to A)$
(xiv) $(A\to B)\to(C\to(A\to B))$
(xv) $((A\to B)\to(A\to C))\to(B\to(A\to B))$
(xvi) $((A\wedge B)\to C)\to(A\to(B\to C))$

where (xi) and (xii) are theorems of RMO^{o}_{+} (see Salto, Robles and Méndez [1999]), (xiii) and (xiv) are provable in $S4_{+}$ (and $S4^{o}_{+}$) and, finally, (xv) and (xvi) are derivable in I_{+} (I^{o}_{+}).

Now, in addition to T1-T10, we have the following theorems (a sketch of the proof is provided at the right of each theorem):

T11. $\neg F$	A1
T12. $\neg B\to((A\to B)\to\neg A)$	A2
T13. $(A\to B)\to(\neg B\to\neg A)$	A3
T14. $(\neg A\wedge\neg B)\to\neg(A\vee B)$	A11, A10, T13
T15. $(\neg A\vee\neg B)\to\neg(A\wedge B)$	A8, T13
T16. $F\to\neg F$	A6
T17. $\neg A\to(A\to\neg A)$	(xi)
T18. $\neg A\to(F\to\neg A)$	(xii)
T19. $A\to\neg F$	(xiii)
T20. $\neg A\to(B\to\neg A)$	(xiv)
T21. $F\to\neg A$	A7

T22. $\neg A\to(A\to\neg B)$	T21
T23. $A\to(\neg A\to\neg B)$	A2, A7
T24. $(\neg A\vee\neg B)\to(A\to\neg B)$	A7, T22
T25. $(A\vee\neg B)\to(\neg A\to\neg B)$	A7, T23
T26. $(A\to(B\to\neg C))\leftrightarrow((A\wedge B)\to\neg C)$	(iv), (xvi)
T27. $(A\to(B\to\neg C))\leftrightarrow((A\to B)\to(A\to\neg C))$	T2, (xv)
T28. $(A\wedge\neg A)\to\neg B$	T23, T27
T29. $((A\vee\neg B)\wedge\neg A)\to\neg B$	T25, T27

T11-T15 are theorems of $T^{o}_{+,F}$**,** T11-T18 are theorems of $RMO^{o}_{+,F}$, T11-T20 are theorems of $S4^{o}_{+,F}$, and finally, T11- T29 are $I^{o}_{+,F}$ theorems (this can readily be seen by inspection of the theorems used in the proofs above).

We note that even *special reductio*

(xvii) $(A\to\neg A)\to\neg A$

i.e.

(xviii) $(A\to(A\to F))\to(A\to F)$

is unprovable. We also remark that although we have some forms of weak contraposition as T12 and T13, still we do not have

(xix) $(A\to\neg B)\to(B\to\neg A)$

i.e.

(xx) $(A\to(B\to F))\to(B\to(A\to F))$

nor

(xxi) $A\to\neg\neg A$

i.e.

(xxii) $A\to((A\to F)\to F)$

Finally, we note that, interestingly, the characteristic intuitionistic CAP axiom (see Robles, Méndez, Salto & Méndez R. [2003])

(xxiii) $F\to(A\to B)$

is not provable.

All these facts follow from the set of matrices in §IX.

V. SEMANTICS

We provide semantics for the logics defined in the previous section.

A $T^o_{+,F}$ model is a quintuple ⟨*K, O, S, R* ,ö⟩ where ⟨*K, O, R,* ö⟩ is a T^o_+ model and *S* a subset of *K* such that $S\cap O\neq\varnothing$. The following clauses are also added:

(v) ($a\leq b$ & *a*ö*F*) ⇒ *b*ö*F*

(vi) *a*ö*F* iff $a\notin S$

ö$_{T^o_{+,F}}$*A* (*A* is $T^o_{+,F}$ valid) iff *a*ö*A* for all $a\in O$ in all models.

We note that *F* is not valid: let $a\in S\cap O$. Then, *a*û*F*. But $a\in O$. So, û$_{T^o_{+,F}}$*A*.

$E^o_{+,F}$ models, $R^o_{+,F}$ models etc. are similarly defined with respect to E^o_+ models, R^o_+ models etc. From now on, let us refer by S^o_+ to any of the positive logics T^o_+, E^o_+, R^o_+, RMO^o_+, $S4^o_+$ or I^o_+. And let $S^o_{+,F}$ be any of the corresponding definitionally extended logics defined in §IV. We have:

THEOREM V.1 (*Semantic consistency of* $S^o_{+,F}$). If ð$_{S^o_{+,F}}$ *A*, then ö$_{S^o_{+,F}}$ *A*.
Proof. Immediate given the semantic consistency of S^o_+.

VI. COMPLETENESS

We sketch a proof of the completeness of $S^o_{+,F}$. The proof is essentially Henkin style in character. That is, we show that for each non-theorem there is a canonical point that does not contain it in the canonical model.

VI.1.*Definition of the canonical model for* $T^o_{+,F}$, $E^o_{+,F}$, $R^o_{+,F}$, *and* $RMO^o_{+,F}$

The $T^o_{+,F}$ canonical model is the structure ⟨K^C, O^C, R^C, S^C, ö^C⟩ where:

- K^C is the set of all prime theories
- O^C is the set of all prime regular theories
- R^C is defined by: R^Cabc iff ($A\rightarrow B\in a$ and $A\in b$) ⇒ $B\in c$
- S^C is the set of all prime consistent theories
- ö^C is defined by: *a*ö^C*A* iff $A\in a$

a is closed under adjunction: if $A\in a$ and $B\in a$, then $A\wedge B\in a$.

a is closed under provable entailment: if ð$_{T^o_{+,F}}$$A\rightarrow B$ and $A\in a$, then $B\in a$.

a is a theory: a is closed under adjunction and provable entailment.

a is prime: if $A\vee B\in a$, then $A\in a$ or $B\in a$.

a is regular: if ð$_{T^o_{+,F}}$*A*, then $A\in a$.

a is consistent: $F \notin a$.

a is non-null: a contains at least one formula.

Now, the $E^{o}_{+,F}$, $R^{o}_{+,F}$ and $RMO^{o}_{+,F}$ canonical model are defined similarly (just change the definition of regular theory according to the notion of theorem in the respective logic).

VI.2 *Definition of the canonical model for $S4^{o}_{+,F}$ and $I^{o}_{+,F}$.*

The $S4^{o}_{+,F}$ canonical model is the structure $\langle K^C, O^C, R^C, S^C, ö^C\rangle$ where R^C and $ö^C$ are defined exactly as in §VI.1 and K^C, O^C and S^C are as in §VI.1 but with this only difference: the theories in these items are now *non-null*[1]. The $I^{o}_{+,F}$ canonical model is defined similarly .

VI.3. *Preliminary lemmas*

We shall employ the following lemmas:

LEMMA VI.1 If x is a theory such that $A \notin x$, there is some $y \in K^C$ such that $x \subseteq y$ and $A \notin y$.

LEMMA VI.2 $a \leq^C b$ iff $a \subseteq b$.

Proof of this lemmas can be found in, e.g., Robles, Méndez, Salto & Méndez R. [2003].

THEOREM VI.1 The canonical model is in fact a model

In the completeness proof of S^{o}_{+} it is shown that the positive canonical model $\langle K^C, R^C, O^C, ö^C\rangle$ is indeed a model. Therefore, in order to prove that the same holds for the canonical $S^{o}_{+,F}$ model, we only have to prove

(a) $S^C \cap O^C ⊠ \varnothing$

and

(b) If canonically read, clauses (v)-(vi) hold.

Proof of (a) $S^C \cap O^C ⊠ \varnothing$: As $û_{S^{o}_{+,F}}\ F$ (see §4), by Theorem V.1 $ú_{S^{o}_{+,F}} F$, i.e., $F \notin S^{o}_{+,F}$. Since $S^{o}_{+,F}$ is a theory, Lemma VI.1 applies and there is some $x \in K^C$ such that $S^{o}_{+,F} \subseteq x$ and $F \notin x$. Thus, x is consistent ($x \in S^C$) and $x \in O^C$.

Proof of (b) Canonical clauses (v)-(vi) hold: By Lemma VI.2 clauses (v) and (vi) are

(v) $(a \subseteq b \text{ and } F \in a) \Rightarrow F \in b$

(vi) $F \in a$ iff $a \notin S^C$

which obviously hold.

We now prove

THEOREM VI.2 (*Completeness of* $S^{o}_{+,F}$) If ö$_{S^{o}_{+,F}}A$, then ð$_{S^{o}_{+,F}}A$.

Proof. Suppose ú$_{S^{o}_{+,F}}A$, i.e., $A\notin S^{o}_{+,F}$. Given that $S^{o}_{+,F}$ is a theory, there is some $x\in K^{C}$ such that $S^{o}_{+,F}\subseteq x$ and $A\notin x$. But x is regular. So, $x\in O^{C}$. As the canonical model is a model, xû^{C}A. Therefore, û$_{S^{o}_{+,F}}A$ (A is not valid).

VII. THE TRUTH CONSTANT *t*

Consider the propositional language of §II extended with the falsity constant F. Now, we add to this language the truth constant t and to $S^{o}_{+,F}$ the axiom

A13. t

and the rule *necessitation* (nec.)

ð$A \Rightarrow$ ð$t\rightarrow A$

We note that

$(t\rightarrow(B\rightarrow C))\rightarrow(B\rightarrow C)$

and

$A\rightarrow t$

could now replace CAP assertion and the K rule to axiomatize $E^{o}_{+,F,t}$ and $S4^{o}_{+,F,t}$ (cfr. Méndez [1987]).

VII.1. *Semantics*

A $T^{o}_{+,F,t}$ model is defined exactly as a $T^{o}_{+,F}$ model but with the addition of the clauses

(vii) ($a\leq b$ and aö t)$\Rightarrow$ böt

(viii) aöt iff $(\exists x\in O)\ x\leq a$

$E^{o}_{+,F,t}$ models, $R^{o}_{+,F,t}$ models, etc are similarly defined with respect to $E^{o}_{+,F}$ models, $R^{o}_{+,F}$ models, etc.

VII.2. *Semantic consistency*

It is clear that in order to prove the semantic consistency of $S^{o}_{+,F,t}$ we just have to prove that A13 is valid and that nec. preserves validity. We shall employ the following lemma:

LEMMA VII.1 ($a\leq b$ and *a*ö*A*) $\Rightarrow$ *b*ö*A*.

This lemma generalizes clause (i) of §2 to all wff and its proof is an easy induction on the length of *A* (use P2 and clauses (v), (vii)).

LEMMA VII.2 A13 is valid.

Proof. Let $x\in O$. By P1, $x\leq x$. So, *x*ö*t*, i.e., ö$_{S^{o}_{+,F,t}}$*t*.

LEMMA VII.3 Nec. preserves validity

Proof. Suppose (for reductio) ö*A* and û$t\rightarrow A$. Then, for some $x\in O$, $y,z\in K$ we have *Rxyz*, *y*ö*t*, *z*û*A*. As *y*ö*t*, $u\leq y$ with $u\in O$. Now, *A* is valid, so *u*ö*A*. By lemma VII.1, *y*ö*A*. By d1, $y\leq z$. Consequently, *z*ö*A* which is impossible.

VII.3. *Completeness*

It is clear that in order to prove the completeness of $S^{o}_{+,F,t}$ we just have to prove that the canonical clauses (vii) and (viii) hold. Clause (vii) is immediate by lemma VI.2 and clause (viii) is proved as follows:

Proof.

(a) Suppose $x\leq^{C}a$ ($x\in O^{C}$). By Lemma VI.2 $x\subseteq a$. Then, obviously $t\in a$ (*x* contains all theorems).

(b) Suppose $t\in a$. We prove: if ð$_{S^{o}_{+,F,t}}$*A*, then $A\in a$. Suppose ð$_{S^{o}_{+,F,t}}$*A*. By nec., ð$_{S^{o}_{+,F,t}}$$t\rightarrow A$. So, $A\in a$ ($t\in a$). Therefore, $a\in O^{C}$. In consequence, ($\exists x\in O^{C}$) $x\subseteq a$. By Lemma VI.2, ($\exists x\in O^{C}$) $x\leq^{C}a$.

VIII. THE RELATIONSHIP BETWEEN *t* AND *F*

The propositional constants are interpreted in standard relevance logics at least as strong as logic of relevance R as follows. The constant *t* represents the conjunction of all truths and the constant *F* the disjunctions of all falsehoods (see, e.g., Anderson and Belnap [1975]). The constant *t* is still the conjunction of all truths in the logics defined in this paper, but the constant *F* is the disjunction of all falsehoods ($F\rightarrow\neg t$) only in $I^{o}_{+,F}$. In particular the following facts are proved. We have *t* and $\neg F$ (A13 and T1, respectively), and $t\rightarrow\neg F$ (by A1 and nec.) in all logics. But $\neg F\rightarrow t$ (A1 and the K rule) only in $S4^{o}_{+,F,t}$ and in $I^{o}_{+,F,t}$. On the other hand, $F\to\neg t$ (A7) is a theorem of $I^{o}_{+,F,t}$ (not of the other logics) but the converse $\neg t\to F$ is not (see the matrices below)[2].

IX. MATRICES

Consider the following set of matrices:

→	0	1	2	3
0	3	2	2	3
1	3	3	3	3
2	3	2	3	3
3	0	2	2	3

∧	0	1	2	3
0	0	1	2	0
1	1	1	1	1
2	2	1	2	2
3	0	1	2	3

∨	0	1	2	3
0	0	0	0	3
1	0	1	2	3
2	0	2	2	3
3	3	3	3	3

where *t* is assigned the only designated value 3, and *F* the value 0. The following facts are deduced from this set:

a) Any logic satisfied by the set has the CAP. Let us consider any wff $(A\to B)\to C$. If →, *t* and *F* do not appear in *C* assign all the variables in *C* the value 1. Then $v(C)=1$ and, so, $v((A\to B)\to C)=2$.

b) The set satisfies $I^{o}_{+,F,t}$.

c) $F\to(A\to B)$ is not satisfied: assign *A* the value 0, and *B* the value 2.

d) Consider now the same set with 3 as the only designated value and 3 the value assigned to *t* as above, but 1 the value assigned now to *F*. Then, the formulas to follow are not satisfied:

$A\to((A\to F)\to F)$

$(A\to(A\to F))\to(A\to F)$

$(A\to(B\to F))\to(B\to(A\to F))$

$\neg t\to F$

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NOTES

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1 Theories have to be non-null to validate P9 and P10. Then, it is easy to prove (in $S4^{o}_{+,F}$ and $I^{O}_{+,F}$): *a* is non-null iff *a* is regular. Then $O^{C}=K^{C}$. This reason explains that in $S4^{o}_{+,F}$ and $I^{o}_{+,F}$ it is more convenient to define validity with respect to *K*. We did not do so in §5 to keep the the treatment of these logics as general as possible.

2 Clauses (v) and (vii) are introduced to preserve closure under the containtment relation (see Lemma VII.1). These clauses could be dispensed with adding the postulates

P11. ($a\leq b$ and $b\leq c$) $\Rightarrow$ $a\leq c$
P12. ($a\leq b$ and $a\notin S$) $\Rightarrow$ $b\notin S$

Proof is left to the reader.

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