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## **Intuitionistic Propositional Logic with the Converse Ackermann Property<sup>1,2</sup>**

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RESUMEN

En este artículo extendemos el espectro de las lógicas con la Conversa de la Propiedad Ackermann. Definimos y axiomatizamos lógicas proposicionales intuicionistas positivas, lógicas subintuicionistas y lógicas intuicionistas con la Conversa de la Propiedad Ackermann. Especialmente interesante es la versión de la negación intuicionista propia de algunos de estos sistemas. Presentamos semánticas de tipo relacional ternario para cada una de las lógicas que estudiamos en este trabajo.

ABSTRACT

The range of propositional logics with the Converse Ackermann Property is proved to be wider than currently assumed. Subintuitionistic, positive intuitionistic and intuitionistic propositional logics with the Converse Ackermann Property are defined and axiomatized. A particular version of intuitionistic negation is involved in some of these systems. Complete ternary relational semantics are offered for all logics studied in the paper.

### I. INTRODUCTION

Consider any propositional logic  $L$  with an implication connective  $\rightarrow$  and a propositional falsity constant  $F$ .  $L$  has the *Converse Ackermann Property* (C.A.P.) if all formulas of the form  $(A \rightarrow B) \rightarrow C$  are unprovable whenever  $C$  contains neither  $\rightarrow$  nor  $F$ . The property is named after the “Ackermann Property”, which is prevalently considered to be a necessary property of any logic of entailment ( $L$  has the *Ackermann Property* if all the formulas of the form  $A \rightarrow (B \rightarrow C)$  are unprovable whenever  $A$  contains neither  $\rightarrow$  nor  $F$ ).

The study of both Ackermann Property and its converse begins with Ackermann [1956 and 1958]. In Anderson & Belnap [1975, §8], Anderson, Belnap & Dunn [1992, §45] C.A.P. and A.P. are defined. Concerning the semantical study of these properties, in Meyer & Routley [1972] “Ackermann groupoids” are firstly defined and used to algebraize a number of relevance logics. Further fruitful applications of these structures may be found in Dunn & Meyer, [1997], Restall [2000] and Kowalski & Ono [2001].

Generally (and syntactically) speaking, logics with the C.A.P. are characterized by the absence of assertion, i.e.,

$$A \rightarrow ((A \rightarrow B) \rightarrow B) \quad (i)$$

and contraction, i.e.,

$$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B) \quad (ii)$$

though restricted versions of both theses can be present.

Now, the C.A.P. is interesting from at least two different points of view: (1) If  $L$  is a logic with the C.A.P., non-necessitive propositions are not derivable from necessitive ones ( $A$  is *necessitive* if  $A$  is of the form  $\Box B$ ) (see Anderson & Belnap [1975] and Restall [2000]). (2) Logics with the C.A.P. are the natural bridge between contractionless (see the foundational Ono & Komori [1985] or the overview Kowalski & Ono [2001]) and contraction logics.

The problem concerning which systems do possess C.A.P. is first posed in Anderson & Belnap [1975, §8.12]. We summarize the current state of the art in the following results: (a) In Méndez [1987] a number of positive logics with the C.A.P. in the spectrum delimited by T-W and J (Ticket entailment without contraction and intuitionistic logic) are defined. (b) In Méndez [1988] these logics are endowed with a sort of semiclassical negation.

(a)-(b) offer a partial solution to the problem, since (i) only relevance subintuitionistic logics are considered, and (ii) only subintuitionistic negation completions are identified as C.A.P. bearers. The aim of this paper is to overcome these limitations by defining propositional intuitionistic logic with the C.A.P.

Let us indicate which are the two main problems we face. First, once negation is introduced with the definition  $\neg A =_{\text{def}} A \rightarrow F$  (along the lines of Johansson [1936]), standard intuitionistic theorems seem unavailable. For example, weak double negation becomes in this setting an instance of assertion ( $A \rightarrow ((A \rightarrow F) \rightarrow F)$ ), but assertion is C.A.P. incompatible. Hence, we shall have to show how to impose additional constraints on  $F$  to define intuitionistic negation. On the other hand, concerning semantics, both Ackermann grupoids and standard ternary frames (see Dunn & Meyer [1997] for an illustrative presentation) should be modified, being intuitionistic logic non-relevant. Our approach adopts properly modified ternary frames. Remarkably, the definition of validity is altered: the set of designated worlds (points, states) is deleted and validity is defined through the set of all possible worlds (points, states).

The structure of the paper is as follows. In §§2,3 we define positive intuitionistic logic without contraction and assertion. In §§4,5 we extend this

system to define positive intuitionistic logic with the C.A.P. In §§6,7,8 intuitionistic negation is added, C.A.P. and semantic consistency are proved. Finally, in §9 we prove completeness. When needed, we shall recall results from Méndez [1987] and Méndez [1988], where the semantics used are different to the models here presented.

## II. POSITIVE INTUITIONISTIC LOGIC WITHOUT CONTRACTION AND ASSERTION: $I_+CA$

The positive language consists of a denumerable set of propositional variables and the binary connectives  $\rightarrow$ ,  $\wedge$  and  $\vee$  ( $A$ ,  $B$ ,  $C$ , etc are metalinguistic variables).  $I_+CA$  can be axiomatized with axioms:

- |     |   |                              |
|-----|---|------------------------------|
| A1. | $A \rightarrow (B \rightarrow A)$   |                              |
| A2. | $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$       |                              |
| A3. | $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$       |                              |
| A4. | $(A \wedge B) \rightarrow A$  | $(A \wedge B) \rightarrow B$ |
| A5. | $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$ |                              |
| A6. | $A \rightarrow (A \vee B)$  | $B \rightarrow (A \vee B)$   |
| A7. | $((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$   |                              |
| A8. | $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)$                               |                              |

Rules: *Modus Ponens* [If  $\vdash A$  and  $\vdash A \rightarrow B$ , then  $\vdash B$ ] and *Adjunction* [if  $\vdash A$  and  $\vdash B$ , then  $\vdash A \wedge B$ ].

## III. SEMANTICS FOR $I_+CA$

Given a pair  $\langle K, R \rangle$  where  $K$  is a non-empty set and  $R$  a ternary relation on  $K$ , let us define the binary relation  $\leq$  and the quaternary relation  $R^2$  by, for every  $a, b, c, d \in K$ .

- |     |   |
|-----|---|
| d1) | $a \leq b$ iff $(\exists x \in K) Rxab$                   |
| d2) | $R^2abcd$ iff $(\exists x \in K)(Rabx \text{ and } Rxcd)$ |

An  $I_+CA$  model is a triple  $\langle K, R, \models \rangle$  where  $K$  is a non-empty set,  $R$  is a ternary relation on  $K$  satisfying the following conditions for every  $a, b, c, d \in K$ ,

- P1.  $a \leq a$   
P2.  $a \leq b$  and  $Rbcd \Rightarrow Racd$   
P3.  $R^2abcd \Rightarrow (\exists x \in K)(Rbcx \text{ and } Raxd)$   
P4.  $R^2abcd \Rightarrow (\exists x \in K)(Racx \text{ and } Rbx d)$   
P5.  $Rabc \Rightarrow a \leq c$

Finally,  $\models$  is a valuation relation from  $K$  to the sentences of  $I_+CA$  satisfying the following conditions for all formulas  $p, A, B$  and point  $a \in K$ :

- i)  $a \models p$  and  $a \leq b \Rightarrow b \models p$   
ii)  $a \models A \vee B$  iff  $a \models A$  or  $a \models B$   
iii)  $a \models A \wedge B$  iff  $a \models A$  and  $a \models B$   
iv)  $a \models A \rightarrow B$  iff for all  $b, c \in K$ ,  $(Rabc \text{ and } b \models A) \Rightarrow c \models B$

$A$  is *valid* in  $I_+CA$  iff  $a \models A$  for all  $a \in K$  in all models. It is easy to prove similarly as in Méndez and Salto [2000] that a formula  $A$  is  $I_+CA$  valid iff  $A$  is a theorem of  $I_+CA$ .

#### IV. POSITIVE INTUITIONISTIC LOGIC WITH THE C.A.P.: $I_+^\circ$

$I_+^\circ$  is axiomatized adding to  $I_+CA$  axioms:

- A9.  $A \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow C))$   
A10.  $(A \rightarrow (A \rightarrow (B \rightarrow C))) \rightarrow (A \rightarrow (B \rightarrow C))$

REMARK 1. A2 (or A3) is not independent.

REMARK 2. Call any formula  $A$  *implicative* iff  $A$  is of the form  $B \rightarrow C$ . A9 and A10 are (i) and (ii) of Introduction restricted to the case in which  $B$  is an implicative formula.

#### V. SEMANTICS FOR $I_+^\circ$

Models for  $I_+^\circ$  are defined similarly to  $I_+CA$  models but with the addition of the following postulates:

- P6.  $R^2abcd \Rightarrow R^2bacd$   
P7.  $R^2abcd \Rightarrow R^3abbcd$

where  $R^3$  is defined by

$$R^3abcde =_{\text{def}} \exists x \exists y (Rabx \text{ and } Rxcy \text{ and } Ryde)$$

We call P1-P7 the *canonical postulates*. We may note that P3 (or P4) is redundant in this setting.

It is easy to prove along the lines of Méndez [1987] that a formula  $A$  is  $I_+^0$ -valid iff  $A$  is a  $I_+^0$ -theorem.

## VI. ADDING INTUITIONISTIC NEGATION: THE LOGIC $I^0$

We add to the sentential language upon which the positive logics are based the propositional falsity constant  $F$  and we define  $\neg A =_{\text{def}} A \rightarrow F$ . Then, the logic  $I^0$  (propositional intuitionistic logic with the C.A.P.) is defined by supplementing  $I_+^0$  with the axioms:

$$\begin{aligned} \text{A11.} \quad & A \rightarrow ((A \rightarrow F) \rightarrow F), \text{ i.e., } A \rightarrow \neg\neg A \\ \text{A12.} \quad & (A \rightarrow (A \rightarrow F)) \rightarrow (A \rightarrow F), \text{ i.e., } (A \rightarrow \neg A) \rightarrow \neg A \\ \text{A13.} \quad & F \rightarrow (A \rightarrow B) \end{aligned}$$

Remarkably, if the standard intuitionistic negation axiom  $F \rightarrow A$  were introduced instead of A13, then the resulting system would not have the C.A.P. However, the following are theorems of  $I^0$ :

$$\begin{aligned} \text{T1.} \quad & F \rightarrow \neg A \\ \text{T2.} \quad & (A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A) \\ \text{T3.} \quad & (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A) \\ \text{T4.} \quad & (A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A) \\ \text{T5.} \quad & (A \rightarrow \neg B) \rightarrow ((A \rightarrow B) \rightarrow \neg A) \\ \text{T6.} \quad & A \rightarrow (\neg A \rightarrow \neg B) \\ \text{T7.} \quad & \neg A \rightarrow (A \rightarrow \neg B) \\ \text{T8.} \quad & (A \wedge \neg A) \rightarrow \neg B \\ \text{T9.} \quad & \neg(A \wedge \neg A) \\ \text{T10.} \quad & \neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B) \\ \text{T11.} \quad & (\neg A \vee \neg B) \rightarrow \neg(A \wedge B) \\ \text{T12.} \quad & (A \vee B) \rightarrow \neg(\neg A \wedge \neg B) \\ \text{T13.} \quad & (A \wedge B) \rightarrow \neg(\neg A \vee \neg B) \\ \text{T14.} \quad & (A \vee \neg B) \rightarrow (\neg A \rightarrow \neg B) \\ \text{T15.} \quad & (A \wedge B) \rightarrow \neg(A \rightarrow \neg B) \\ \text{T16.} \quad & (A \rightarrow B) \rightarrow \neg(A \wedge \neg B) \\ \text{T17.} \quad & ((A \vee \neg B) \wedge \neg A) \rightarrow \neg B \end{aligned}$$

## VII. CONVERSE ACKERMANN PROPERTY

Consider the following set of matrices where  $F$  is assigned the value 0 and 2 is the only designated value

$\rightarrow$	0	1	2
0	2	0	2
1	2	2	2
2	0	0	2

$\wedge$	0	1	2
0	0	1	0
1	1	1	1
2	0	1	2

$\vee$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

This set verifies  $I^0$  but falsifies  $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$  only when  $v(A) = 0$ ,  $v(B) = 1$  and  $A \rightarrow ((A \rightarrow B) \rightarrow B)$  only when  $v(A) = 2$ ,  $v(B) = 1$ . We show that  $I^0$  has the C.A.P. Let  $(A \rightarrow B) \rightarrow C$  a wff in which  $C$  contains nor  $\rightarrow$  neither  $F$ . Assign all variables in  $C$  the value 1. Then  $v((A \rightarrow B) \rightarrow C) = 0$ .

Note that A11 and A12 are, of course, not independent in logics with assertion and contraction. However, they are not derivable from  $I^0+$  and A13: assign  $F$  the value 1.

VIII. SEMANTICS FOR  $I^0$ 

A  $I^0$  model is a quadruple  $\langle K, S, R, \models \rangle$  where  $S$  is a non-empty subset of  $K$  and  $\langle K, R, \models \rangle$  is an  $I_+^0$  model such that the conditions below are satisfied:

- P8.  $(Rabc \text{ and } c \in S) \Rightarrow \exists x(x \in S \text{ and } Rbax)$   
P9.  $(Rabc \text{ and } c \in S) \Rightarrow \exists x \exists y (Rabx \text{ and } Rxb y \text{ and } y \in S)$   
P10.  $a \perp S \Rightarrow (\text{not-}Rabc \text{ or } c \models A) \text{ (for any wff } A)$ .

and the relation  $\models$  satisfies in addition the following clause

- v)  $a \models F$  iff  $a \notin S$

A formula  $A$  is  $I^0$ -valid iff  $a \models A$  for all  $a \in K$  in all models.

We sketch a proof of semantic consistency (semantic soundness of  $I^0$  relative to the semantics of  $I^0$  models). First we prove

LEMMA 8.1. If  $a \leq b$  and  $a \models A$ , then  $b \models A$ .

*Proof.* By induction on the length of  $A$ , using P2 in the case of the conditional.

LEMMA 8.2.  $\models_r A \rightarrow B$  iff for all  $a \in K$  in all models,  $a \models A \Rightarrow a \models B$ .

*Proof.* P1, d1 and Lemma 8.1.

Now, we can prove:

**THEOREM 8.1.** (Semantic consistency of  $I^o$  (soundness of  $I^o$ )) If  $\vdash_{I^o} A$ , then  $\models_{I^o} A$ .

*Proof.* Since all theorems are conditional formulas, we can use Lemma 8.2. to immediately render A4-A8 and the rules. A1, A2, A3, A9 and A10 are proved using, respectively, P5, P3, P4, P6 and P7 (see Méndez (1987)). Finally, A11, A12 and A13 use, respectively, P8, P9 and P10. We illustrate the procedure by proving the validity of A12. Suppose arguing by reductio ad absurdum that  $a \models A \rightarrow (A \rightarrow F)$  and  $a \not\models A \rightarrow F$  for some  $a \in K$  in some model (Lemma 8.2). By definitions, there are  $b, c \in K$  such that  $Rabc$ ,  $b \delta A$  and  $c \not\models F$ . As  $c \in S$  (clause (v)),  $Rbx$  and  $Rxy$  for some  $x \in K, y \in S$  (P9). Hence,  $x \models A \rightarrow F$  (since  $Rbx$ ,  $a \delta A \rightarrow (A \rightarrow F)$  and  $b \models A$ ) and finally  $y \models F$  (because  $Rxy$ ,  $x \models A \rightarrow F$  and  $b \models A$ ) which is impossible,  $y$  being a member of  $S$ .

## IX. COMPLETENESS OF $I^o$

We begin by recalling some definitions. A set of formulas  $a$  is a *theory* if it is closed under Adjunction and also has the property that if  $A \in a$  and  $A \rightarrow B$  is a provable formula then,  $B \in a$ . Let  $K^T$  be the set of all theories, and  $R^T$  the ternary relation on  $K^T$  defined as follows: for every  $a, b, c \in K^T$ ,  $R^T abc$  iff for all formulas  $A, B$  such that  $A \rightarrow B \in a$  and  $A \in b$ , it holds that  $B \in c$ . A theory  $b$  is *prime* just in case  $A \in b$  or  $B \in b$  whenever  $A \vee B \in b$ , and *consistent* iff no negation of a theorem does not belong to  $b$ . A theory is *regular* iff contains all  $I^o$  theorems and *null* iff no formula belongs to it. Now, let  $K^C$  be the set of all prime non-null theories and  $S^C$  the set of all consistent theories, and  $R^C$  the restriction of  $R^T$  to  $K^C$ . Further, let  $\models^C$  be defined for any wff  $A$  and  $a \in K^C$  as follows:  $a \models^C A$  iff  $A \in a$ . Then, the quadruple  $\langle K^C, S^C, R^C, \models^C \rangle$  is called the  $I^o$  canonical model.

$I^o$  completeness will follow from the facts contained in the next series of lemmas. The first one states the equivalence between regular and non-null theories, as proved using A1:

**LEMMA 9.1.** If  $b$  is any non null theory, then  $b$  is regular

The following proposition shows that for each non-null theory lacking a formula there is a non-null prime theory extending it which lacks the same formula:

LEMMA 9.2. Let  $A$  be a wff,  $b$  a non-null element in  $K^T$  and  $A \notin b$ . Then,  $A \notin x$  for some  $x \in K^C$  such that  $b \subseteq x$ .

*Proof.* By Zorn's Lemma there is a maximal theory  $x$  without  $A$  including  $b$ . If  $x$  is not prime, then for some wffs  $B, C$ ,  $B \vee C \in x$ ,  $B \notin x$ ,  $C \notin x$ . Put  $[B, x] = \{E: \exists D(D \in x \text{ and } \delta_F(B \wedge D) \rightarrow E)\}$ . Define  $[C, x]$  similarly. Clearly both  $[B, x]$  and  $[C, x]$  are non-null theories strictly including  $x$ . Since  $x$  is maximal,  $A \in [B, x]$  and  $A \in [C, x]$ , which implies  $A \in x$ , and this is impossible.

LEMMA 9.3. Let  $a, b, c \in K^T$  such that  $a$  is non-null,  $c$  is prime and  $R^T abc$ . Then there is  $x \in K^C$  such that  $a \subseteq x$  and  $R^T xbc$ .

*Proof.* By Zorn's Lemma there is a maximal theory  $x$  including  $a$  such that  $R^T xbc$ . Suppose  $x$  is not prime and define  $[A, x]$  and  $[B, x]$  as in Lemma 9.2 for formulas  $A, B$  such that  $A \vee B \in x$  but  $A \notin x$ ,  $B \notin x$ . Since both  $[A, x]$  and  $[B, x]$  are theories strictly including  $x$ , we have  $\text{not-}R^T[A, x]bc$  and  $\text{not-}R^T[B, x]bc$ . But then it is easily shown that  $c$  is not prime, which is impossible.

LEMMA 9.4. Let  $a, b, c \in K^T$  such that  $b$  is non-null,  $c$  is prime and  $R^T abc$ . Then there is  $x \in K^C$  such that  $b \subseteq x$  and  $R^T axc$ .

*Proof.* Similar to that of Lemma 9.3.

LEMMA 9.5. Let  $a, b$  non-null theories. The set  $x = \{B: \exists A(A \rightarrow B \in a \text{ and } A \in b)\}$  is a non-null theory such that  $R^T abx$ .

*Proof.* It is easy to prove that  $x$  is closed under Adjunction and provable entailment. Obviously,  $R^T abx$ . Next, let  $\vdash_F A$  and  $B \in b$ . By A1,  $\vdash B \rightarrow A$  so a fortiori,  $B \rightarrow A \in a$  (Lemma 9.1). Therefore,  $A \in x$ , i.e.,  $x$  is regular.

The following lemma shows that the relation  $\cdot^C$  of the canonical model is the set inclusion relation.

LEMMA 9.6.  $a \leq^C b$  iff  $a \subseteq b$

*Proof.* Suppose  $a \leq^C b$ . By d1,  $R^C xab$  for some  $x \in K^C$ . As  $A \rightarrow A \in x$  (Lemma 9.1), if  $A \in a$ , then  $A \in b$ , i.e.,  $a \subseteq b$ . Suppose now  $a \subseteq b$ . Since  $a \in K^T$ , clearly  $R^T Paa$ . Hence,  $R^T Pab$ . By Lemma 9.3, there is some  $x \in K^C$  such that  $R^C xab$ , which, by d1, just is  $a \leq^C b$  as required.

It is easy to prove the following fact using A11:

LEMMA 9.7.  $F \in a$  iff  $a$  is inconsistent

LEMMA 9.8. The canonical  $\models^C$  is a valuation relation satisfying conditions (i)-(v) [§2, §7]



*Proof.* (i) is trivial by Lemma 9.6, (ii) and (iii) are immediate given  $I^0$  and properties of members of  $K^C$ , (v) follows from Lemma 9.7 and (iv) from right to left is obvious. So, we prove (iv) from left to right. Assume  $a \not\equiv^C A \rightarrow B$ . We show that there are  $x, y \in K^C$  such that  $R^C axy$ ,  $x \equiv^C A$  and  $y \not\equiv^C B$ . Define  $b = \{C: \vdash_P A \rightarrow C\}$ . It is simple to verify that  $b$  is a non-null theory such that  $A \in b$  ( $\vdash_P A \rightarrow A$ ). Define now (Lemma 9.5) the non-null theory  $c = \{C: \exists D(D \rightarrow C \in a \text{ and } D \in b)\}$  such that  $R^T abc$ . Further, we show  $B \notin c$ . Suppose arguing by reductio ad absurdum that  $B \in c$ . Then there is a formula  $C \in b$  such that  $C \rightarrow B \in a$ . Therefore,  $\vdash_P A \rightarrow C$  and, so,  $A \rightarrow B \in a$ , i.e.,  $a \equiv^C A \rightarrow B$ , contradicting the hypothesis. Now Lemma 8.2 applies, and there is some  $y \in K^C$  such that  $c \subseteq y$  and  $B \notin y$ . By definitions,  $R^T aby$ . Hence, Lemma 9.4 grants there is some  $x \in K^C$  such that  $b \subseteq x$  and  $R^C axy$  with  $A \in x$  ( $A \in b$ ).

LEMMA 9.9. The canonical postulates hold in the  $I^0$  canonical model.

*Proof.* P1 holds by Lemma 9.6; P2 by definition of  $R^C$  and Lemma 9.6; P3, P4, P5, P6 and P7 follow from, respectively, A2, A3, A1, A9 and A10 using lemmas 9.2-9.6. We exemplify the proof concerning the validity of P4. We show:

PROPOSITION: Let  $a, b, c$  be non-null elements in  $K^T$  and  $d \in K^C$ . Moreover, assume  $R^{T2}abcd$ . Then, there is some  $x \in K^C$  such that  $R^T acx$  and  $R^T bxd$ .

*Proof.* Suppose  $R^{T2}abcd$ , that is,  $R^T abx$  and  $R^T xcd$  for some non-null element  $x$  in  $K^T$  (Lemma 9.5). We have to prove that there is some  $x \in K^C$  such that  $R^T acx$  and  $R^T bxd$ . Define (Lemma 9.5) the non-null theory  $z = \{B: \exists A(A \rightarrow B \in a \text{ and } A \in c)\}$  with  $R^T acz$ . Deduce now  $R^T bzd$  using A3. By Lemma 9.4,  $R^T bxd$  with  $z \subseteq x$  and  $x \in K^C$ . By  $R^T acz$  and definitions,  $R^T acx$ .

The canonical postulate P4, i.e.,  $R^{C2}abcd \Rightarrow \exists x(R^C acx \text{ and } R^C bxd)$  immediately follows.

Finally, the validity of P8, P9 and P10 hold by respectively the three lemmas that follow.

LEMMA 9.10. Let  $a, b, c$  be non-null members of  $K^T$ ,  $c$  a consistent element in  $K^T$  and  $R^T abc$ . Then, there is some  $x$  in  $S^C$  such that  $c \subseteq x$  and  $R^T bax$ .

*Proof.* Define (cfr. Lemma 9.5) the non-null theory  $y = \{B: \exists A(A \rightarrow B \in b \text{ and } A \in a)\}$ . Thus:  $R^T bay$ . We prove  $y$  is consistent. Suppose it is not. Then,  $F \in y$  (Lemma 9.7). By definition of  $y$ , let  $A \in a$  such that  $A \rightarrow B \in b$ . By A11,  $(A \rightarrow F) \rightarrow F \in a$ . Given that  $R^T abc$ ,  $F \in c$ , contradicting the hypothesis. Then, apply Lemma 9.2.

LEMMA 9.11. Given non-null  $a, b \in K^T$ ,  $c$  a consistent theory and  $R^T abc$ , then there is some  $x \in K^C$  and  $y \in S^C$  such that  $R^T abx$  and  $R^T xby$ .

*Proof.* Suppose  $R^T abc$ ,  $a, b, c$  non-null theories and  $c$  being consistent. Define the non-null theories

$$u = \{B: \exists A(A \rightarrow B \in a \text{ and } A \in b)\}$$

$$w = \{B: \exists A(A \rightarrow B \in u \text{ and } A \in b)\}$$

thus:  $R^T abu$  and  $R^T ubw$ . We prove first  $w$  consistent. Suppose it is not. Then,  $F \in w$  (Lemma 9.7). By definition of  $w$ ,  $B \rightarrow F \in u$  ( $B \in b$ ). By definition of  $u$ ,  $A \rightarrow (B \rightarrow F) \in a$  ( $A \in b$ ). As  $(A \rightarrow (B \rightarrow F)) \rightarrow ((A \wedge B) \rightarrow F)$  is a  $P^o$  theorem,  $(A \wedge B) \rightarrow F \in a$ . But since  $A \wedge B \in b$  ( $A, B \in b$ ) and  $R^T abc$ ,  $F \in c$ , which is impossible  $c$  being consistent.

Therefore, we have  $u, w \in K^T$  (with  $w$  consistent). Hence, Lemma 9.2. applies and in consequence there is some  $y$  in  $S^C$  such that  $w \subseteq y$  and  $R^T uby$ . Now, by Lemma 9.3 there is some  $x$  in  $K^C$  such that  $u \subseteq x$  and  $R^T xby$ . As  $R^T abu$  (and  $u \subseteq x$ ),  $R^T abx$  as required.

LEMMA 9.12. Let  $a$  be an inconsistent theory. Then, for any non-null theory  $b$  and prime  $c$  in  $K^T$ , not- $R^T abc$  or  $B \in c$ , for any wff  $B$ .

*Proof.* Suppose  $F \in a$  (Lemma 9.7),  $Rabc$ ,  $A \in b$  and  $B$  any wff. By A13,  $A \rightarrow B \in a$ , hence  $B \in c$ . That is, if  $a$  is inconsistent, for any non-null theory  $b$  and prime theory  $c$  either  $Rabc$  does not obtain or else  $c$  is degenerate (a theory is *degenerate* iff any wff belongs to it).

Now, canonical postulates P8, P9 and P10, i.e.,

$$(R^C abc \text{ and } c \in S^C) \Rightarrow \exists x(x \in S^C, c \subseteq x \text{ and } R^C bax)$$

$$(R^C abc \text{ and } c \in S^C) \Rightarrow \exists x \exists y(y \in S^C, R^C abx \text{ and } R^C xby)$$

and

$$a \in S^C \Rightarrow \forall x \forall y(\text{not-}R^C axy \text{ or } y \text{ is degenerate})$$

are immediate from, respectively, lemmas 9.10, 9.11 and 9.12.

Three last lemmas before the completeness theorem:

LEMMA 9.13. If  $\nVdash P^o A$ , then there is some  $x \in K^C$  such that  $A \notin x$ .

*Proof.*  $P^o$  is the minimal regular theory such that  $A \notin P^o$ . Then, use Lemma 9.2.

LEMMA 9.14.  $S^C$  is non-empty

*Proof.* As  $\not\vdash_P F$ , we have, by Theorem 8.1,  $\not\vdash_P F$ . Then apply Lemma 9.13.

LEMMA 9.15. The  $P$  canonical model is indeed a  $P$  model.

*Proof.* By Lemma 9.14.,  $S^C$  is non-empty. Lemmas 9.10-9.12 show that the canonical postulates P.8, P.9 and P.10 hold in the canonical model. Thus, by Lemma 9.9 we have that the  $P$  canonical model is an  $P$  model.

Finally, Lemmas 9.13. and 9.15. prove:

THEOREM 9.1. (Completeness of  $P$ ) If  $\vDash_P A$ , then  $\vdash_P A$

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#### NOTES

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