Language as a Geometry in Wittgenstein's *Tractatus*

Leo K. C. Cheung, Hong Kong

1. Die Bildhaftigkeit

In TLP 4.011, while admitting that propositions expressed by the phonetic notation, or the alphabet, just like the written notes of a piece of music, do not seem at first sight to be pictures of what they represent, the *Tractatus* insists that those 'sign-languages' (that is, the phonetic notation and the written musical notes) prove to be pictures of what they represent (that is, our speech and the piece of music, respectively) 'even in the ordinary sense'. (TLP 4.016 also says that 'alphabetic script developed out of [hieroglyphic script] without losing what was essential to depiction'.) So, contrary to the view of some commentators (e.g. Pears 1987, 115-121), instead of making an analogy here, the *Tractatus* holds that a proposition is a picture *literally*. How can a proposition be a picture literally?

The *Tractatus* explains in TLP 4.012 that '…a proposition of the form *'aRb'* strikes us as a picture. In this case the sign is obviously a likeness of what is signified'. The likeness between propositions and their senses is constituted by what is called the '*Bildhaftigkeit*' or pictorial character (TLP 4.013). The *Tractatus* goes on to characterize the *Bildhaftigkeit* via the internal relation of depiction, as well as the inner similarity, between different sign-languages (or different expressions) of a piece of music, which is also claimed to be holding between language and the world:

> A gramophone record, the musical idea, the written notes, and the sound-waves all stand to one another in the same internal relation of depicting that holds between language and the world…
>
> There is a general rule by means of which the musician can obtain the symphony from the score, and which makes it possible to derive the symphony from the groove on the gramophone record, and, using the first rule, to derive the score again. That is what constitutes the inner similarity between these things which seem to be constructed in such entirely different ways. And that rule is the law of projection which projects the symphony into the language of musical notation. It is the rule for translating this language into the language of gramophone records. (TLP 4.014-4.0141)

Note that the general rule is also called 'the law of projection'. What by means of which, say, the symphony can be derived from the score is presumably a method of projection or a specific rule determined by the general rule. In general, the *Bildhaftigkeit* of all our modes of expression consists in the logic of depiction (TLP 4.015). Therefore, for the *Tractatus*, the followings are equivalent:

[a] A thing can be a picture of another thing.

[b] There is a specific rule of depiction (determined by the general rule of depiction), or a method of projection, by means of which one thing can be derived from another thing.

[c] There is an inner similarity between two things (or one thing is a likeness of another thing).

2. The Possibility of a Generalization

The *Tractatus* also takes the *Bildhaftigkeit* as an agreement in form. For in order for a picture to be able to depict, it must have a form – its pictorial form – in common with reality (TLP 2.17-2.174 and 2.18-2.182). That is, a thing can be a picture of another thing if and only if they have a form in common. But it is not clear why there must be an agreement in form if one is a picture of another and vice versa. Wittgenstein says later in *Philosophical Grammar* that such move is misleading, and what he is in effect doing in the *Tractatus* is extending or generalizing the concept of 'having in common' and taking it equivalent to the concept of projection:

> …what I said really boils down to this: that every projection must have something in common with what is projected no matter what is the method of projection. But that only means that I am only here extending the concept of 'having in common' and am taking it equivalent to the general concept of projection. So I am only drawing attention to the possibility of a generalization (which is of course can be very important). (PG, p.163)

In addition, he told Friedrich Waismann in 1931 that the *Tractatus* inherited the concept of a picture 'from two sides: first from a drawn picture, second from the picture of a mathematician, which already is a general concept' (LWVC, p.185). Actually, the connection of the notion of a picture to projective geometry is already indicated in the *Tractatus*. For example, as already mentioned, the general rule in TLP 4.0141 is also called 'the law of projection'. In TLP 3.1-3.13, Wittgenstein regards the perceptible sign of a proposition as a projection of a possible situation and talks about the method of projection. All these suggest an illuminating way of reading the account of a picture which reflects what Wittgenstein is *in effect* doing in the *Tractatus*.

Two points can be drawn here. First, instead of taking the *Bildhaftigkeit* as an *agreement* in form, the *Tractatus* should be read as taking the concept of having a form in common equivalent to that of projection. This means that the following is also equivalent to any of [a]-[c]:

[d] Two things have the same (pictorial) form in common.

In order to understand the Tractarian notion of a picture, an independent characterization of one of the key notions in [a]-[d] is needed. Fortunately, such characterization can be found in the next point.

The second point is that the Tractarian notion of a picture is a generalization of the mathematical concept of a picture, as exemplified in the case of projective geometry (cf., Rhees 1996, 4). The key is that the move of making the concept of having in common equivalent to that of projection consists in adopting 'that every projection must have something in common with what is projected no matter what is the method of projection', and that the something here is a form of projection. A form of projection is an invariant under whatever method of projection (or under all specific rules of projection). The concept of projective geometry adopted by the *Tractatus* is then as follows: Projective geometry is constituted by the specific rules of projection and the invariants (forms of projection), subject to this constraint:

> Two figures have the same invariant (form of projection) in common if and only if there is a specific rule of projection according to which one figure can be projected onto another.

The constraint, as one will see, is what guarantees the consistency of the notion of, and thus the possibility of, those invariants. In the case of depiction or [d], *a (pictorial) form can be characterized as an invariant under all specific rules of depiction*. In the particular case of language, a propositional form is to be characterized as an invariant under all specific rules of language. The mutual equivalence of [a]-[d] can then be formulated as follows:

> There is a specific rule of depiction according to which one thing can be derived from another thing (or one thing is a likeness of another, or one thing is a picture of another) if and only if the two things have the same invariant (pictorial form) in common.

What conditions must language satisfy such that the above characterization makes sense and is consistent?

3. Language, Geometry and the Erlanger Programm

The *Tractatus*, of course, does not identify language with projective geometry but rather holds that language is, just like projective geometry, a geometry. How can language be a geometry? To see this, suppose that X is a set, and G is a set of rules (hereafter 'G-rules), each of which sends, or maps, members of X to members of X. A G-invariant, or a G-form, may be defined as what is invariant under the application of all G-rules. The pair <G, X> is said to determine a geometry if:

> [*] Two members of X have the same G-form (G-invariants) in common if and only if there is a rule in G according to which one member is mapped to another member.

What are the conditions that <G, X> must satisfy in order for the notions in [*] to be well-defined and consistent? Define the relation ~ as follows: A~B if and only if there is a G-rule according to which A is mapped to B. It is easy to prove that the condition [*] is satisfied if and only if ~ is an equivalence relation. In the latter case, a G-invariant (G-form) can be identified with the equivalence classes which partition X, and this would guarantee that the notion of a G-form is well-defined. Moreover, it can also be proven that ~ is an equivalence relation if and only if G equipped with the composition operation constitutes a (mathematical) group. (For details, see any good textbook on Algebra or Yaglom 1988, 112-116.) Let me explain the notion of a group here. G is equipped with the composition operation if:

> For any members A, B and C of X, if there is a G-rule α mapping A to B and a G-rule β mapping B to C, then there is a G-rule mapping A to C.

In this case, the G-rule mapping A to C may be denoted by '$\beta\circ\alpha$', or simply '$\beta\alpha$'. The operation $\circ$ here is called 'composition'. G equipped with the composition operation is called 'a group' if these four conditions are satisfied:

(1) For any G-rules α and β, $\beta\alpha$ is a G-rule.

(2) For any G-rules α, β and γ, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

(3) There is a G-rule, denoted by 'ε', such that $\alpha\varepsilon = \varepsilon\alpha = \alpha$, for any G-rule α.
ε is said to be the unit of G.
In this case, ε is the G-rule mapping an member of X to the same member.

(4) For any G-rule α, there is a G-rule β such that $\beta\alpha = \alpha\beta = \varepsilon$.
In this case, ψ is said to be the inverse of α and may be denoted by 'α^{-1}'.

To employ popular mathematical terms, a G-rule is a transformation from X to X, and G is a transformation group if it satisfies the above four conditions. A geometry can then be defined as *what is determined by a transformation group with the invariants (forms) under all the transformations of the group as its objects.* Projective geometry is a particular case here, where G is the set of all specific rules of projection and the forms of projection are the G-invariants. So is language, the *Tractatus* would say.

Language is another specific case with X being the set of all facts (TLP 2.141 and 3.14), G the set of all specific rules of language and pictorial forms the G-invariants. My claim that the *Tractatus* holds this is supported by the passages in TLP 4.014-4.0141 already quoted above. Recall that the general rule (or the law of projection) enables the derivation of the symphony from the score (via a specific rule), the symphony from the groove (via another specific rule) and, using the first (specific) rule, the score from the symphony. Let the two specific rules involved be ϕ and ψ, respectively. ϕ sends the score to the symphony, and is also the same specific rule sending the symphony back to the score, while ψ sends the groove to the symphony. In this case, $\phi\phi$ sends the score to the score itself and thus is the unit-rule ε. Also, ϕ is the inverse-rule of ϕ itself. This suggests that, first, the composition of specific rules is possible and results in another specific rule, that is, (1) holds. Second, there is a unit-rule, that is, ε exists and (3) holds. Third, every specific rule has an inverse-rule, that is, (4) holds. The *Tractatus* does not indicate if (2) is accepted as well. But, given that the other three are accepted, it is reasonable to believe that it does. Thus, the *Tractatus* in effect takes the set of all specific rules of depiction, or language, as a transformation group. Pictorial forms, or propositional forms, are then invariants under a transformation group. It is then proven that, for the *Tractatus*, language is a geometry with the invariants under the relevant transformation group as its objects.

In 1872, the mathematician Felix Klein announced what was later called 'Das Erlanger Programm' in his famous inauguration lecture (Klein 1921, 460-497). Its kernel is the definition of a geometry as the study of the invariants under a transformation group (ibid., 463). It was found later that the 'Erlanger Programm' can characterize geometries like projective geometry, euclidean geometry and some non-euclidean geometries but not those like algebraic geometry and Riemannian geometry. (For an excellent discussion of the 'Erlanger Programm', see Yaglom, 1988, 111-124.) Nevertheless, it is still safe to say that the study of the invariants under a transformation group is a geometry. The 'Erlanger Programm' was very influential amongst mathematicians and physicists, as well as philosophers who were interested in what was going on in the areas of mathematics and science, in late nineteenth century and early twentieth century. Russell has been greatly influenced by Klein, as one can see from his *An Essay on the Foundations of Mathematics* (Russell, 1897), *The Principles of Mathematics* (Russell 1937, 435-436) and his preface to the *Tractatus* (TLP, p.xi). It is not sure if

Wittgenstein knew the 'Erlanger Programm'. But, as the discussion in this paper suggests, it is very likely that the *Tractatus* at least has been influenced by views originated from the 'Erlanger Programm'.

References

Klein, F. 1921 'Vergleichende Betrachtungen über neuere geometrische Forschungen (Das Erlanger Programm.)', in F. Klein, *Gesammelte Mathematische Abhandlungen*, *Erster Band*, Berlin: Julius Springer, 460-497.

Pears, D. 1987 *The False Prison Volume One*, Oxford: Clarendon.

Rhees, R. 1996 'Miss Anscombe on the *Tractatus*', in R. Rhees, *Discussions of Wittgenstein*, Bristol: Thoemmes.

Russell, B. 1897 *An Essay on the Foundations of Mathematics*, Cambridge: Cambridge University Press.

Russell, B. 1937 *The Principles of Mathematics*, London: George Allen & Unwin.

Wittgenstein, L. 1974 *Tractatus Logico-Philosophicus* [TLP], D. F. Pears and B. F. McGuinness (trans.), London: RKP.

Wittgenstein, L. 1978 *Philosophical Grammar* [PG], R. Rhees (ed.), A. Kenny (trans.), Oxford: Blackwell.

Wittgenstein, L. 1979 *Ludwig Wittgenstein and the Vienna Circle* [LWVC], B. McGuinness (ed.), J. Schulte and B. McGuinness (trans), Oxford: Blackwell.

Yaglom, I. M. 1988 *Felix Klein and Sophus Lie*, H. Grant and A. Shenitzer (ed.), S. Sossinsky (trans.), Boston: Birkhäuser.